arXiv:0705.3362v1 [math.AP] 23 May 2007

Title: Convergence to Equilibrium for the Cahn-Hilliard Equation with Wentzell Boundary Condition

Author:
Hao Wu
School of Mathematical Sciences
Fudan University
Shanghai 200433, P.R. China

Mailing address:
Hao Wu
School of Mathematical Sciences
Fudan University
Han Dan Road No. 220, Shanghai, 200433,
P.R. China
Email: haowufd@yahoo.com

Convergence to Equilibrium for the Cahn-Hilliard Equation with Wentzell Boundary Condition

Hao Wu
School of Mathematical Sciences
Fudan University
200433 Shanghai, P.R. China
haowufd@yahoo.com

February 1, 2008

Abstract

In this paper we consider the Cahn-Hilliard equation endowed with Wentzell boundary condition which is a model of phase separation in a binary mixture contained in a bounded domain with permeable wall. Under the assumption that the nonlinearity is analytic with respect to unknown dependent function, we prove the convergence of a global solution to an equilibrium as time goes to infinity by means of a suitable Łojasiewicz-Simon type inequality with boundary term. Estimates of convergence rate are also provided.

Keywords: Cahn-Hilliard equation, Wentzell boundary conditions, Łojasiewicz-Simon inequality, convergence to equilibrium.

1 Introduction

This paper is concerned with the asymptotic behavior of the global solution to the following Cahn-Hilliard equation

$$u_t = \Delta\mu, \qquad \text{in } [0,T]\times\Omega \tag{1.1}$$

$$\mu = -\Delta u + f(u), \qquad \text{in } [0,T]\times\Omega, \tag{1.2}$$

subject to Wentzell boundary condition

$$\Delta\mu + b\partial_\nu \mu + c\mu = 0, \qquad \text{on} \;\; [0,T]\times\Gamma, \tag{1.3}$$

the variational boundary condition

$$-\alpha\Delta_{\|}u + \partial_\nu u + \beta u = \frac{\mu}{b}, \qquad \text{on} \;\; [0,T]\times\Gamma, \tag{1.4}$$

and initial datum

$$u(0,x) = \psi_0, \qquad \text{in} \;\; \Omega. \tag{1.5}$$

In above, $0 < T \le \infty$, Ω is a bounded domain in $\mathbb{R}^n$ $(n = 2,3)$ with smooth boundary Γ. α, β, b, c are positive constants. $\Delta_{\|}$ is the Laplace-Beltrami operator on Γ, and ν is the outward normal direction to the boundary.

The Cahn-Hilliard equation arises from the study of spinodal decomposition of binary mixtures that appears, for example, in cooling process of alloys, glass or polymer mixtures (see [1,12,20,28] and the references cited therein). μ is called chemical potential in the literature. The classical Cahn-Hilliard equation is equipped with the following homogeneous Neumann boundary conditions

$$\begin{aligned} \partial_\nu \mu &= 0, \qquad t>0,\; x\in\Gamma, &(1.6)\\ \partial_\nu u &= 0, \qquad t>0,\; x\in\Gamma. &(1.7)\end{aligned}$$

Boundary (1.6) has a clear physical meaning: there cannot be any exchange of the mixture constituents through the boundary Γ which implies that the total mass $\int_\Omega u dx$ is conversed for all time. The boundary condition (1.7) is usually called variational boundary condition which together with (1.6) result in decreasing of the following bulk free energy

$$E_b(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u)\right) dx, \tag{1.8}$$

where $F(s) = \int_0^s f(z)dz$. A typical example in physics for potential F is the so-called 'double-well' potential $F(u) = \frac{1}{4}(u^2-1)^2$.

For the equations (1.1) (1.2) subject to boundary conditions (1.6) (1.7) and initial datum (1.5), extensive study has been made. We refer e.g., to [3, 7, 19, 25, 28, 36] and the references cited therein. In particular, convergence to equilibrium for the global solution in higher space dimension case was proved in [25].

Recently, a new model has been derived when the effective interaction between the wall (i.e., the boundary Γ) and two mixture components are short-ranged (see Kenzler et

al. [12]). In such a situation, it is pointed out in [12] that, the following surface energy functional

$$E_s(u) = \int_\Gamma \left(\frac{\sigma_s}{2} \left|\nabla_\| u\right|^2 + \frac{g_s}{2} u^2 - h_s u \right) dS, \tag{1.9}$$

with $\nabla_\|$ being the covariant gradient operator on Γ (see e.g. [17]), should be added to the bulk free energy $E_b(u)$ to form a total free energy functional

$$E(u) = E_b(u) + E_s(u). \tag{1.10}$$

In above, $\sigma_s > 0$, $g_s > 0$, $h_s \neq 0$ are given constants. Together with the no-flux boundary (1.6) condition, the following dynamical boundary condition is posed in order that the total energy $E(u)$ is decreasing with respect to time:

$$\sigma_s \Delta_\| u - \partial_\nu u - g_s u + h_s = \frac{1}{\Gamma_s} u_t, \qquad t > 0,\ x \in \Gamma. \tag{1.11}$$

We refer to [2, 17, 23, 24, 29] for extensive study for system (1.1)(1.2) with boundary conditions (1.6)(1.11) and initial datum (1.5). In particular, Wu & Zheng [29] proved the convergence to equilibrium for a global solution as time goes to infinity by deriving a new type of Łojasiewicz-Simon inequality with boundary term (see also [2] for a different proof).

Based on the above model, in a quite recent article by Gal [4], the author proposed (1.1)–(1.5) as a variation model which describes phase separation in a binary mixture confined to a bounded region Ω with porous walls. Instead of the no-flux boundary condition (1.6), the Wentzell boundary condition (1.3) is derived from mass conservation laws that include an external mass source (energy density) on boundary Γ. This may be realized, for example, by an appropriate choice of the surface material of the wall, i.e., the wall Γ may be replaced by a penetrable permeable membrane (ref. [4]). Then, the variational boundary condition (1.4) is introduced in order that the system (1.1)–(1.5) tends to minimize its total energy

$$E(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 + F(u) \right) dx + \int_\Gamma \left(\frac{\alpha}{2} \left|\nabla_\| u\right|^2 + \frac{\beta}{2} u^2 \right) dS. \tag{1.12}$$

Namely,

$$\frac{d}{dt} E(u(t)) = - \int_\Omega |\nabla \mu|^2 dx - \frac{c}{b} \int_\Gamma \mu^2 dS \le 0. \tag{1.13}$$

For more intensive discussions, we refer to [4–6].

In [4], the existence and uniqueness of global solution to problem (1.1)-(1.5) has been proved by adapting the approach in [24]. Later, in [5], the same author studied the problem in a further way that he obtained the existence and uniqueness of a global

solution to the problem under more general assumptions than those in [4]. He showed that the global solution defines a semiflow on certain function spaces and also proved the existence of an exponential attractor with finite dimension.

Then a natural question is: whether the global solution of system (1.1)-(1.5) will converge to an equilibrium as time goes to infinity? This is just the main goal of this paper. Moreover, we shall provide estimates for the rate of the convergence (in higher order norm).

Remark 1.1. *Without loss of generality, in the following text, we set positive constants* b, c, α, β *to be* 1. *In this paper, we simply use* $\|\cdot\|$ *for the norm on* $L^2(\Omega)$ *and equip* $H^1(\Omega)$ *with the equivalent norm*

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 dx + \int_\Gamma u^2 dS\right)^{1/2}. \tag{1.14}$$

Before stating our main result, first we make some assumptions on nonlinearity f.

(**F1**) $f(s)$ is analytic in $s \in \mathbb{R}$.

(**F2**)

$$|f(s)| \le C(1 + |s|^p), \quad \forall s \in \mathbb{R},$$

where $C \ge 0$, $p > 0$ and $p \in (0, 5)$ for $n = 3$.

(**F3**)

$$\liminf_{|s|\to\infty} f'(s) > 0.$$

Remark 1.2. *Assumption (**F1**) is made so that we are able to derive an extended Łojasiewicz-Simon inequality to prove our convergence result. Assumption (**F2**) implies that the nonlinear term has a subcritical growth. Assumption (**F3**) is some kind of dissipative condition. (**F3**) is supposed in [4, 5] to obtain the existence and uniqueness of global solution to the evolution problem* (1.1)–(1.5). *Moreover, (**F3**) together with (**F2**) enable us to prove the existence result for stationary problem* (1.18) *by variational method (see Section 3). It's easy to check that the nonlinearity* $f(u) = u^3 - u$ *corresponding to the most important physical potential* $F(u) = \frac{1}{4}(u^2 - 1)^2$ *satisfies all the assumptions stated above.*

Let V be the Hilbert space which, as introduced in [4], is the completion of $C^1(\Omega)$ with the following inner product and the associated norm:

$$(u, v)_V = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Gamma \left(\nabla_\| u \cdot \nabla_\| v + uv\right) dS, \quad \forall\, u, v \in V. \tag{1.15}$$

The main result of this paper is as follows.

Theorem 1.1. *Let (**F1**)–(**F3**) be satisfied. For any initial datum $u_0 \in V$, the solution $u(t,x)$ to problem (1.1)–(1.5) converges to a certain equilibrium $\psi(x)$ in the topology of $H^3(\Omega) \cap H^3(\Gamma)$ as time goes to infinity, i.e.,*

$$\lim_{t\to+\infty} \left(\|u(t,\cdot)-\psi\|_{H^3(\Omega)} + \|u(t,\cdot)-\psi\|_{H^3(\Gamma)}\right) = 0. \tag{1.16}$$

Moreover, we have the following estimate for the rate of convergence:

$$\|u-\psi\|_{H^3(\Omega)} + \|u-\psi\|_{H^3(\Gamma)} + \|u_t\|_V \le C(1+t)^{-\theta/(1-2\theta)}, \qquad t \ge 1. \tag{1.17}$$

Here, $C \ge 0$, $\psi(x)$ is an equilibrium to problem (1.1)–(1.5), i.e., a solution to the following nonlinear boundary value problem:

$$\begin{cases} -\Delta\psi + f(\psi) = 0, & x \in \Omega, \\ -\Delta_{\|}\psi + \partial_\nu\psi + \psi = 0, & x \in \Gamma, \end{cases} \tag{1.18}$$

and $\theta \in (0, \frac{1}{2})$ is a constant depending on $\psi(x)$.

Before giving the detailed proof of Theorem 1.1, let's first recall some related results in the literature. The study of asymptotic behavior of solutions to nonlinear dissipative evolution equations has attracted a lot of interests of many mathematicians for a long period of time. Unlike in 1-d case (see [16, 35]), the situation in higher space dimension case can be quite complicated. On one hand, the topology of the set of stationary solutions can be non-trivial and may form a continuum. On the other hand, a counterexample has been given in [22] for a semilinear parabolic equation saying that even the nonlinear term being C^∞ cannot ensure the convergence to a single equilibrium (see also [21]). In 1983 Simon in [26] proved that for a semilinear parabolic equation if the nonlinearity is analytic in unknown function u, then convergence to equilibrium for bounded global solutions holds. His idea relies on generalization of the Łojasiewicz inequality (see [13–15]) for analytic functions defined in finite dimensional space $\mathbb{R}^m$. Since then, Simon's idea has been applied to prove convergence results for many evolution equations, see e.g., [8–11, 25] and the references cited therein. To the best of our knowledge, most previous work are concerned with evolution equations subject to homogeneous Dirichlet or Neumann boundary conditions.

Our problem (1.1)–(1.5) has the following features. The first boundary condition (1.3) is Wentzell boundary condition which involves the time derivative of u; the second boundary condition (1.4) for u has a mixed type since it also involves the chemical potential μ. It turns out that for the corresponding elliptic operator, it yields a non-homogeneous

boundary condition. The Łojasiewicz-Simon inequality for homogeneous boundary conditions in the literature fails to apply. As a result, a non-trivial modification is required to treat the present problem. We succeed in deriving an extended Łojasiewicz-Simon type inequality involving boundary term, with which we able to show the convergence result (for other applications, see [29–31, 33]). Besides, by delicate energy estimates and constructing proper differential inequalities, we are able to obtain the estimates for the convergence rate (in higher order norm). This in some sense improves the previous result in the literature (see for instance [8, Theorem 1.1]) and can apply to other evolution equations (ref. [31, 32]).

The rest part of this paper is organized as follows: In Section 2 we introduce the functional settings and present some known results on existence and uniqueness of global solution and uniform compactness obtained in [4, 5]. In Section3 we study the stationary problem. Section 4 is devoted to prove an extended Łojasiewicz-Simon inequality with boundary term. In the final Section 5 we give the detailed proof of Theorem 1.1.

2 Preliminaries

We shall use the functional settings introduced in [4, 5].

For $u \in C(\overline{\Omega})$, we identify u with the vector $U = (u\big|_\Omega, u\big|_\Gamma) \in C(\Omega) \times C(\Gamma)$. We define $\mathcal{H} = L^2(\Omega) \oplus L^2(\Gamma)$ to be the completion of $C(\overline{\Omega})$ with respect to the following norm

$$\|u\|_{\mathcal{H}} = \left(\|u\|^2 + \|u\|^2_{L^2(\Gamma)}\right)^{\frac{1}{2}}. \tag{2.1}$$

For any $g \in \mathcal{H}$, consider the elliptic boundary value problem

$$\begin{cases} -\Delta u = g, & \text{in } \Omega, \\ \partial_\nu u + u = g, & \text{on } \Gamma. \end{cases} \tag{2.2}$$

We can associate it with the following bilinear form on $H^1(\Omega)$:

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla u dx + \int_\Gamma uv dS, \tag{2.3}$$

for all $u, v \in H^1(\Omega)$. Then it defines a strictly positive self-adjoint unbounded operator $A : D(A) = \{u \in H^1(\Omega) | Au \in \mathcal{H}\} \to \mathcal{H}$ such that

$$\langle Au, v \rangle_{\mathcal{H}} = a(u, v), \quad \forall\ u \in D(A),\ v \in H^1(\Omega). \tag{2.4}$$

Then by Lax-Milgram theorem, it follows that the operator A is a bijection from $D(A)$ into $\mathcal{H}$ and $A^{-1} : \mathcal{H} \to \mathcal{H}$ is a linear, self-adjoint and compact operator on $\mathcal{H}$ (see [5, Section 2] or [4, Section 4]). In other words, for any $g \in \mathcal{H}$, $A^{-1}g$ is the unique solution to (2.2).

We can then consider the *weak energy space* X endowed with the following norm,

$$\|v\|_{\mathrm{X}}^2 = \|A^{-1/2}v\|_{\mathcal{H}}^2 = \langle A^{-1}v, v\rangle_{\mathcal{H}}, \qquad \forall\ v \in \mathcal{H}. \tag{2.5}$$

It follows that

$$\big\langle u, v\big\rangle_{\mathrm{X}} = \big\langle u, A^{-1}v\big\rangle_{\mathcal{H}}, \quad \forall\ u \in H^1(\Omega),\ v \in \mathrm{X}. \tag{2.6}$$

In particular, for all $v \in \mathrm{X}$ and $u = A^{-1}v$

$$\|v\|_{\mathrm{X}}^2 = \langle A^{-1}v, v\rangle_{\mathcal{H}} = \langle u, Au\rangle_{\mathcal{H}} = a(u, u). \tag{2.7}$$

For more detailed discussions, we refer to [4,5].

The existence and uniqueness of global solution to (1.1)-(1.5) has been obtained in [4,5]. The results in [5, Section 3,4] and [4, Section 4] in particular imply that

Theorem 2.1. *Let **(F1)**–**(F3)** be satisfied. For any initial datum $u_0 \in V$, problem* (1.1)-(1.5) *admits a unique global solution $u(t,x)$ which defines a global semiflow on V. Moreover, $u(t,x)$ belongs to C^∞ for $t > 0$.*

The total free energy

$$E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u)\right) dx + \int_\Gamma \left(\frac{1}{2}\left|\nabla_\| u\right|^2 + \frac{1}{2}u^2\right) dS. \tag{2.8}$$

where $F(s) = \int_0^s f(z)dz$, serves as a Lyapunov functional for problem (1.1)–(1.5). In other words, for the smooth solution u to problem (1.1)–(1.5), we have

$$\frac{d}{dt}E(u) + \int_\Omega |\nabla \mu|^2 + \int_\Gamma |\mu|^2 dS = 0. \tag{2.9}$$

Uniform bounds for the solution which yield the relative compactness in $H^3(\Omega)\cap H^3(\Gamma)$ can be seen from [5, Proposition 3.3, Theorem 3.5], here we state the result without proof.

Lemma 2.1. *Let **(F1)**–**(F3)** hold and $\gamma \in [0, 1/2)$. Then, for any initial datum $u_0 \in V$, the solution of* (1.1)–(1.5) *satisfies the following dissipative estimates, namely, for any $\delta > 0$, there hold*

$$\|u(t)\|_{H^{3+\gamma}(\Omega)}^2 + \|u(t)\|_{H^{3+\gamma}(\Gamma)}^2 \le C_\delta, \quad t \ge \delta > 0, \tag{2.10}$$

and (ref. [5, (3.34)]),

$$\|u_t\|_{H^1(\Omega)}^2 + \|u_t\|_{H^1(\Gamma)}^2 \le C_\delta, \quad t \ge \delta > 0, \tag{2.11}$$

where $C_\delta > 0$ depends only on $\|u_0\|_V$ and δ.

For any initial datum $u_0 \in V$, the ω-limit set of u_0 is defined as follows:

$$\omega(u_0) = \{\psi(x) \mid \ \exists \{t_n\} \text{ such that } u(t_n, x) \to \psi(x) \in \ V, \text{ as } t_n \to +\infty\}.$$

Then we have

Lemma 2.2. *For any $u_0 \in V$, the ω-limit set of u_0 is a compact connected subset in $H^3(\Omega) \cap H^3(\Gamma)$. Furthermore,*
(i) $\omega(u_0)$ is invariant under the nonlinear semigroup $S(t)$ defined by the solution $u(x,t)$, i.e, $S(t)\omega(u_0) = \omega(u_0)$ for all $t \geq 0$.
(ii) $E(u)$ is constant on $\omega(u_0)$. Moreover, $\omega(u_0)$ consists of equilibria.

Proof. Since our system has a continuous Lyapunov functional $E(u)$, the conclusion of the present lemma follows from Lemma 2.1 and the well-known results in the dynamical system (e.g. [28, Lemma I.1.1]). Thus, the lemma is proved. □

3 Stationary Problem

In this section we study the stationary problem. The stationary problem corresponding to (1.1)–(1.5) is

$$\begin{cases} \Delta\widetilde{\mu} = 0, & x \in \Omega, \\ -\Delta\psi + f(\psi) = \widetilde{\mu}, & x \in \Omega, \\ \partial_\nu\widetilde{\mu} + \widetilde{\mu} = 0, & x \in \Gamma, \\ -\Delta_\| \psi + \partial_\nu\psi + \psi = \widetilde{\mu}, & x \in \Gamma. \end{cases} \tag{3.1}$$

Then it immediately follows that $\widetilde{\mu} = 0$ and the stationary problem is reduced to (1.18).

Lemma 3.1. *Let **(F1)**–**(F3)** be satisfied. Suppose that $\psi \in H^3(\Omega) \cap H^3(\Gamma)$ satisfies (1.18). Then ψ is a critical point of the functional $E(u)$ over V. Conversely, if $\psi \in V$ is a critical point of $E(u)$, then $\psi \in C^\infty$ and it is a classical solution to problem (1.18).*

Proof. The proof is similar to [29, Lemma 2.1]. The C^∞ regularity for solution ψ follows from the elliptic regularity for (1.18) (see e.g., [17, Corollary A.1, Lemma A.2]) and a bootstrap argument. □

Lemma 3.2. *Let **(F1)**–**(F3)** be satisfied. The functional $E(u)$ has at least a minimizer $v \in V$ such that*

$$E(v) = \inf_{u \in V} E(u). \tag{3.2}$$

In other words, problem (1.18) admits at least a classical solution.

Proof. From assumption (**F3**), there exists $\delta_0 > 0$ such that

$$\liminf_{|s|\to+\infty} f'(s) \geq \delta_0. \tag{3.3}$$

Then there exists $N_1 = N_1(\delta_0) > 0$ such that

$$f'(s) \geq \frac{1}{2}\delta_0, \qquad |s| \geq N_1. \tag{3.4}$$

It follows that

$$\liminf_{s\to+\infty} f(s) \geq 1, \quad \liminf_{s\to-\infty} f(s) \leq -1. \tag{3.5}$$

Since $F'(s) = f(s)$, then we can deduce from (3.5) that

$$\liminf_{|s|\to+\infty} F(s) \geq 1. \tag{3.6}$$

Therefore, there exists $N_2 \geq 0$ such that

$$F(s) \geq 0, \qquad |s| \geq N_2. \tag{3.7}$$

This indicates that

$$\int_\Omega F(u)dx = \int_{|u|>N_2} F(u)dx + \int_{|u|\leq N_2} F(u)dx \geq |\Omega| \min_{|s|\leq N_2} F(s) > -\infty. \tag{3.8}$$

$E(u)$ can be written in the form:

$$E(u) = \frac{1}{2}\|u\|_V^2 + \mathcal{F}(u) \tag{3.9}$$

with

$$\mathcal{F}(u) = \int_\Omega F(u)dx. \tag{3.10}$$

It follows that $E(u)$ is bounded from below on V, namely,

$$E(u) \geq \frac{1}{2}\|u\|_V^2 + C_f, \tag{3.11}$$

where $C_f := |\Omega| \min_{|s|\leq N_2} F(s)$. It's easy to see that constant C_f depends only on f and Ω. Therefore, there is a minimizing sequence $u_n \in V$ such that

$$E(u_n) \to \inf_{u\in V} E(u). \tag{3.12}$$

It follows from (3.11) that u_n is bounded in V. It turns out from the weak compactness that there is a subsequence, still denoted by u_n, such that u_n weakly converges to v in V. Thus, $v \in V$. We infer from the Sobolev imbedding theorem that the imbedding $V \subset H^1(\Omega) \hookrightarrow L^\gamma(\Omega)$ ($1 \leq \gamma < \frac{n+2}{n-2}$) is compact. As a result, u_n strongly converges to v in $L^\gamma(\Omega)$. It turns out from the assumption (**F2**) that $\mathcal{F}(u_n) \to \mathcal{F}(v)$. Since $\|u\|_V^2$ is weakly lower semi-continuous, it follows from (3.12) that $E(v) = \inf_{u\in V} E(u)$.

The proof is completed. $\square$

4 Extended Łojasiewicz-Simon Inequality

In what follows, we prove a suitable version of extended Łojasiewicz-Simon inequality required in the proof of our main result.

Let ψ be a critical point of $E(u)$. We consider the following linearized operator

$$L(v)h \equiv -\Delta h + f'(v+\psi)h \tag{4.1}$$

with the domain being defined as follows.

$$Dom(L(v)) = \{h \in H^2(\Omega) \cap H^2(\Gamma) : -\Delta_{\|} h + \partial_\nu h + h \mid_\Gamma = 0\} := \mathcal{D}. \tag{4.2}$$

The equivalent norm on $\mathcal{D}$ is

$$\|u\|_{\mathcal{D}} := \|u\|_{H^2(\Omega)} + \|u\|_{H^2(\Gamma)}. \tag{4.3}$$

It's obvious that $\mathcal{D} \subset L^2(\Omega)$ is dense in $L^2(\Omega)$, and $L(v)$ maps $\mathcal{D}$ into $L^2(\Omega)$. In analogy to [29, Lemma 2.3], we know that $L(v)$ is self-adjoint.

Associated with $L(0)$, we define the bilinear form $b(w_1, w_2)$ on V as follows.

$$b(w_1, w_2) = \int_\Omega (\nabla w_1 \cdot \nabla w_2 + f'(\psi) w_1 w_2) dx + \int_\Gamma \left(\nabla_{\|} w_1 \cdot \nabla_{\|} w_2 + w_1 w_2\right) dS \tag{4.4}$$

Then, the same as for the usual second order elliptic operator, $L(0) + \lambda I$ with $\lambda > 0$ being sufficiently large is invertible and its inverse is compact in $L^2(\Omega)$. It turns out from the Fredholm alternative theorem that $Ker(L(0))$ is finite-dimensional. It is well known that

$$Ran(L(0)) = (Ker(L(0))^*)^{\perp}. \tag{4.5}$$

Thus, we infer from the fact that $L(0)$ is a self-adjoint operator that

$$Ran(L(0)) = (Ker(L(0)))^{\perp}, \qquad Ran(L(0)) \oplus Ker(L(0)) = L^2(\Omega). \tag{4.6}$$

Next we introduce two orthogonal projections Π_K and Π_R in $L^2(\Omega)$, namely, Π_K is the projection onto the kernel of $L(0)$ while Π_R is the projection onto the range of $L(0)$. Then we have the following result.

Lemma 4.1. *For*

$$L(0)w = f_R$$

with $f_R \in L^2(\Omega)$, there exists a unique solution $w_R \in \mathcal{D}$ and the following estimate holds:

$$\|w_R\|_{\mathcal{D}} \le C\|f_R\|. \tag{4.7}$$

Proof. By the Fredholm alternative theorem and the regularity theorem for the elliptic operator (see [17]), we have a function $w \in \mathcal{D}$ such that $L(0)w = f_R$. Moreover w is unique if we require $w \in (KerL(0))^{\perp}$, and (4.7) follows from the elliptic regularity theory. $\square$

Let $\mathcal{L}(v) : \mathcal{D} \to L^2(\Omega)$ be defined as follows:

$$\mathcal{L}(v)w = \Pi_K w + L(v)w. \tag{4.8}$$

Then it follows from the above lemma that $\mathcal{L}(0)$ is bijective and its inverse $\mathcal{L}^{-1}(0)$ is a bounded linear operator from $L^2(\Omega)$ to $\mathcal{D}$.

Lemma 4.2. *There exists a small constant $\beta < 1$ depending on ψ such that for any $v \in \mathcal{D}$, $\|v\|_{H^2(\Omega)} \leq \beta$ and $f \in L^2(\Omega)$,*

$$\mathcal{L}(v)w = f \tag{4.9}$$

admits a unique solution w such that $w \in \mathcal{D}$ and the following estimate holds,

$$\|w\|_{\mathcal{D}} \leq C\|f\|. \tag{4.10}$$

Proof. It follows from the above lemma that $\mathcal{L}(0)$ is bijective and its inverse $\mathcal{L}^{-1}(0)$ is a bounded linear operator from $L^2(\Omega)$ to $\mathcal{D}$. We rewrite (4.9) into the following form:

$$(\mathcal{L}^{-1}(0)(\mathcal{L}(v) - \mathcal{L}(0)) + I)w = \mathcal{L}^{-1}(0)f. \tag{4.11}$$

From the definition, we have $(\mathcal{L}(v) - \mathcal{L}(0))w = (f'(v+\psi) - f'(\psi))w.$

We infer from Sobolev imbedding theorem that for any $\|v\|_{H^2} \leq \beta \ll 1$, there holds

$$\|(f'(v+\psi) - f'(\psi))w\| \leq C\|v\|_{H^2(\Omega)}\|w\|_{\mathcal{D}}. \tag{4.12}$$

Therefore, it follows that when β is sufficiently small, $\mathcal{L}^{-1}(0)(\mathcal{L}(v)-\mathcal{L}(0))$ is a contraction from $\mathcal{D}$ to $\mathcal{D}$:

$$\|\mathcal{L}^{-1}(0)(\mathcal{L}(v) - \mathcal{L}(0))\|_{L(\mathcal{D},\mathcal{D})} \leq \frac{1}{2}. \tag{4.13}$$

By the contraction mapping theorem, (4.11) is uniquely solvable which implies that when $\|v\|_{H^2(\Omega)} \leq \beta$, $\mathcal{L}(v)$ is invertible, and (4.10) holds.

Thus, the lemma is proved. $\square$

Let ψ be a critical point of $E(u)$. Denote $u = v + \psi$ and

$$\mathcal{E}(v) = E(u) = E(v+\psi). \tag{4.14}$$

Let

$$M(v) = -\Delta(v + \psi) + f(v + \psi). \tag{4.15}$$

Then for any $v \in \mathcal{D}$, $M(v) \in L^2(\Omega)$.

First, we prove the following Łojasiewicz-Simon inequality for the homogeneous boundary condition corresponding to the nonhomogeneous one (1.4).

Lemma 4.3. *Let ψ be a critical point of $E(u)$. There exist constants $\theta^* \in (0, \frac{1}{2})$ and $\beta^* \in (0, \beta)$ depending on ψ such that for any $w \in \mathcal{D}$, if $\|w\|_{\mathcal{D}} < \beta^*$, there holds*

$$\|M(w)\| \ \geq \ |\mathcal{E}(w) - E(\psi)|^{1-\theta^*}. \tag{4.16}$$

Proof. Let $\mathcal{N} : \mathcal{D} \mapsto L^2(\Omega)$ be the nonlinear operator defined as follows

$$\mathcal{N}(w) = \Pi_K w + M(w). \tag{4.17}$$

Then $\mathcal{N}(w)$ is differentiable and

$$D\mathcal{N}(w)h = \mathcal{L}(w)h. \tag{4.18}$$

By the result in [18], we know that

Lemma 4.4. *The mapping $L^\infty(\Omega) \ni u \to f(u) \in L^\infty(\Omega)$ is analytic.*

It easily follows from Lemma 2.1, Sobolev imbedding theorem and above lemma that $\mathcal{N}(w)$ is analytic. Since $\mathcal{L}(0)$ is invertible, by the abstract implicit function theorem (for the analytic version see e.g. [34, Corallary 4.37, p.172]), there exist neighborhoods of the origin $W_1(0) \subset \mathcal{D}$, $W_2(0) \subset L^2(\Omega)$ and an analytic inverse mapping Ψ of $\mathcal{N}$ such that $\Psi : W_2(0) \to W_1(0)$ is 1-1 and onto. Besides,

$$\mathcal{N}(\Psi(g)) = g \qquad \forall g \in W_2(0), \tag{4.19}$$

$$\Psi(\mathcal{N}(v)) = v \qquad \forall v \in W_1(0), \tag{4.20}$$

and in analogy to the argument in [27, Lemma 1, pp.75] (see also [10, Lemma 5.4]) we can show that

$$\|\Psi(g_1) - \Psi(g_2)\|_{\mathcal{D}} \ \leq C\|g_1 - g_2\| \qquad \forall g_1, g_2 \in W_2(0), \tag{4.21}$$

$$\|\mathcal{N}(v_1) - \mathcal{N}(v_2)\| \ \leq C\|v_1 - v_2\|_{\mathcal{D}} \qquad \forall v_1, v_2 \in W_1(0). \tag{4.22}$$

Let $\phi_1, ..., \phi_m$ be the orthogonal unit vectors spanning $Ker(L(0))$.
Since Ψ is analytic, it turns out that

$$\Gamma(\xi) := \mathcal{E}\left(\Psi\left(\sum_{i=1}^{m} \xi_i \phi_i\right)\right) \tag{4.23}$$

is analytic with respect to $\xi = (\xi_1, ..., \xi_m)$ with $|\xi|$ sufficiently small such that $\Pi_K w = \sum_{i=1}^{m} \xi_i \phi_i \in W_2(0)$.

With the aid of $\Gamma(\xi)$ which is an analytic function defined in $\mathbb{R}^m$, we are able to apply the Łojasiewicz inequality. By the standard argument (see e.g. [10]), we can show that, there exist constants $\theta^* \in (0, \frac{1}{2})$ and $\beta^* \in (0, \beta)$ depending on ψ such that for any $w \in \mathcal{D}$ with $\|w\|_{\mathcal{D}} < \beta^*$, there holds

$$\|M(w)\| \ \geq \ |\mathcal{E}(w) - \Gamma(0)|^{1-\theta^*}, \tag{4.24}$$

which is exactly (4.16). The details are omitted. □

Now we are in a position to prove the following extended Łojasiewicz-Simon inequality with boundary term.

Lemma 4.5. *Let ψ be a critical point of $E(u)$. Then there exist constants $\theta \in (0, \frac{1}{2})$, $\beta_0 \in (0, \beta)$ depending on ψ such that for any $u \in H^3(\Omega)$, if $\|u - \psi\|_{H^2(\Omega)} < \beta_0$, the following inequality holds,*

$$\|M(v)\| + \| - \Delta_{\|} u + \partial_\nu u + u\|_{L^2(\Gamma)} \ \geq \ |E(u) - E(\psi)|^{1-\theta}. \tag{4.25}$$

Proof. For any $u \in H^3(\Omega)$, let $v = u - \psi$. Then $v \in H^3(\Omega)$.
We consider the following elliptic boundary value problem:

$$\begin{cases} -\Delta w = -\Delta v, & x \in \Omega, \\ -\Delta_{\|} w + \partial_\nu w + w = 0, & x \in \Gamma. \end{cases} \tag{4.26}$$

Since $\Delta v \in L^2(\Omega)$, similar to the previous discussion for $L(0)$, it follows that equation (4.26) admits a unique solution $w \in \mathcal{D}$. From the H^2-regularity for (4.26) (see e.g., [17, Appendix Lemma A.1]), it turns out that

$$\|w\|_{H^2(\Omega)} + \|w\|_{H^2(\Gamma)} \leq C\|\Delta v\| \leq C\|v\|_{H^2(\Omega)}. \tag{4.27}$$

Hence, there exists $\widetilde{\beta} \in (0, \beta)$ such that for $\|v\|_{H^2(\Omega)} < \widetilde{\beta}$ we have

$$\|w\|_{\mathcal{D}} < \beta^*. \tag{4.28}$$

Here β^* is the constant in Lemma 4.3. Thus, (4.16) holds for w.

On the other hand, (4.26) can be rewritten in the following form

$$\begin{cases} -\Delta(w - v) = 0, & x \in \Omega, \\ -\Delta_{\|}(w - v) + \partial_\nu(w - v) + (w - v) = \Delta_{\|} v - \partial_\nu v - v, & x \in \Gamma. \end{cases} \tag{4.29}$$

Again from [17, Appendix Lemma A.1], there holds

$$\|w-v\|_{H^1(\Omega)}+\|w-v\|_{H^1(\Gamma)}\le C\|\Delta_\| v-\partial_\nu v-v\|_{L^2(\Gamma)}. \tag{4.30}$$

By straightforward computation,

$$\begin{aligned}\|M(w)\| &\le \left(\|M(v)\|+C\|v-w\|_{H^1(\Omega)}\right)\\ &\le \left(\|M(v)\|+C\|\Delta_\| v-\partial_\nu v-v\|_{L^2(\Gamma)}\right). \end{aligned}\tag{4.31}$$

Meanwhile, it follows from Newton-Leibniz formula that

$$\begin{aligned} & \mid E(w+\psi)-E(v+\psi)\mid \\ \le & \left|\int_0^1\int_\Omega M(v+t(w-v))(v-w)dxdt\right| \\ & +\left|\int_0^1\int_\Gamma (1-t)(\Delta_\| v-\partial_\nu v-v)(v-w)dSdt\right| \\ \le & C\left(\|M(v)\|+\|\Delta_\| v-\partial_\nu v-v\|_{L^2(\Gamma)}\right)\|\Delta_\| v-\partial_\nu v-v\|_{L^2(\Gamma)} \\ \le & C\left(\|M(v)\|+\|\Delta_\| v-\partial_\nu v-v\|_{L^2(\Gamma)}\right)^2. \end{aligned}\tag{4.32}$$

Since

$$\begin{aligned} & |E(w+\psi)-E(\psi)\mid^{1-\theta^*} \\ \ge & \mid E(v+\psi)-E(\psi)\mid^{1-\theta^*}-\mid E(w+\psi)-E(v+\psi)\mid^{1-\theta^*}, \end{aligned}\tag{4.33}$$

and $0<\theta^*<\frac{1}{2}$, $2(1-\theta^*)-1>0$, then we infer from (4.31)–(4.33) that

$$C(\|M(v)\|+\|\Delta_\| v-\partial_\nu v-v\|_{L^2(\Gamma)})\ \ge |E(u)-E(\psi)|^{1-\theta^*}.$$

Taking $\varepsilon\in(0,\theta^*)$ and $\beta_0\in(0,\widetilde{\beta})$, such that for $\|v\|_{H^2}<\beta_0$,

$$\frac{1}{C}\mid E(v+\psi)-E(\psi)|^{-\varepsilon}\ge 1. \tag{4.34}$$

Let $\theta=\theta^*-\varepsilon\in(0,\frac{1}{2})$, then for $\|v\|_{H^2}<\beta_0$, there holds

$$\|M(v)\|+\|\Delta_\| v-\partial_\nu v-v\|_{L^2(\Gamma)}\ \ge |E(u)-E(\psi)|^{1-\theta}, \tag{4.35}$$

which is exactly (4.25) by the definition of v. □

5 Convergence to equilibrium and convergence rate

After the previous preparations, we now proceed to finish the proof of Theorem 1.1.

Part I. Convergence to Equilibrium

From the previous results, there exists an increasing sequence $\{t_n\}_{n\in\mathbb{N}}$, $t_n \to +\infty$ and $\psi \in \omega(u_0)$ such that

$$\lim_{t_n\to+\infty} \|u(t_n,x)-\psi(x)\|_{H^3(\Omega)} = 0. \tag{5.1}$$

On the other hand, it follows from (2.9) that $E(u)$ is decreasing in time. We now consider all possibilities.

(1). If there is a $t_0 > 0$ such that at this time $E(u) = E(\psi)$, then for all $t > t_0$, we deduce from (2.9) that $\|\mu(t)\|_{H^1(\Omega)} \equiv 0$. On the other hand, it follows from (1.1) (1.3) that

$$\begin{cases} -\Delta\mu = -u_t, & x\in\Omega, \\ \partial_\nu\mu + \mu = -u_t, & x\in\Gamma. \end{cases} \tag{5.2}$$

Then by (2.7) we have

$$\|u_t\|_{\mathrm{X}}^2 = \int_\Omega \mu u_t dx + \int_\Gamma \mu u_t dS = \|\mu\|_{H^1(\Omega)}^2. \tag{5.3}$$

This implies that $\|u_t\|_{\mathrm{X}} \equiv 0$, i.e., u is independent of t for all $t > t_0$. Since $u(x,t_n) \to \psi$, then (1.16) holds.

(2). If for all $t > 0$, $E(u) > E(\psi)$, and there is $t_0 > 0$ such that for all $t \geq t_0$, $v = u - \psi$ satisfies the condition of Lemma 4.5, i.e., $\|u-\psi\|_{H^2(\Omega)} < \beta_0$, then for the constant $\theta \in (0, \frac{1}{2})$ in Lemma 4.5, we have

$$-\frac{d}{dt}(E(u)-E(\psi))^\theta = -\theta(E(u)-E(\psi))^{\theta-1}\frac{dE(u)}{dt}. \tag{5.4}$$

From (1.2), $M(v) = \mu$. Then it follows from (1.14)(2.9) and Lemma 4.5 that

$$-\frac{d}{dt}(E(u)-E(\psi))^\theta \geq \theta\frac{\|\nabla\mu\|^2 + \|\mu\|_{L^2(\Gamma)}^2}{\|\mu\| + \|\mu\|_{L^2(\Gamma)}} \geq C_\theta\|\mu\|_{H^1(\Omega)}. \tag{5.5}$$

Integrating from t_0 to t,

$$(E(u)-E(\psi))^\theta + C_\theta\int_{t_0}^t \|\mu\|_{H^1(\Omega)}d\tau \leq (E(u(t_0))-E(\psi))^\theta. \tag{5.6}$$

Since, $E(u(t)) - E(\psi) \geq 0$, we have

$$\int_{t_0}^t \|\mu\|_{H^1(\Omega)}d\tau < +\infty, \qquad \forall\, t \geq t_0. \tag{5.7}$$

Thus, (5.3)(5.7) imply that for all $t \geq t_0$,

$$\int_{t_0}^t \|u_t\|_{\mathrm{X}}d\tau < +\infty, \tag{5.8}$$

which easily yields that as $t \to +\infty$, $u(t,x)$ converges in X. Since the orbit is compact in $H^3(\Omega) \cap H^3(\Gamma)$, we can deduce from uniqueness of limit that (1.16) holds.

(3). It follows from (5.1) that for any $\varepsilon \in (0, \beta_0)$, there exists $N \in \mathbb{N}$ such that when $n \geq N$,

$$\| u(t_n, \cdot) - \psi \|_{\mathrm{X}} \leq \|u(t_n, \cdot) - \psi\|_{H^3(\Omega)} < \frac{\varepsilon}{2}, \tag{5.9}$$

$$\frac{1}{C_\theta}(E(u(t_n)) - E(\psi))^\theta < \frac{\varepsilon}{2}. \tag{5.10}$$

Define

$$\bar{t}_n = \sup\{\ t > t_n \mid \| u(s, \cdot) - \psi \|_{H^2(\Omega)} < \beta_0,\ \forall s \in [t_n, t]\}. \tag{5.11}$$

(5.1) and continuity of the orbit in $H^2(\Omega)$ yield that $\bar{t}_n > t_n$ for all $n \geq N$.

Then there are two possibilities:

(i). If there exists $n_0 \geq N$ such that $\bar{t}_{n_0} = +\infty$, then from the previous discussions in (1) and (2), (1.16) holds.

(ii) Otherwise, for all $n \geq N$, we have $t_n < \bar{t}_n < +\infty$, and for all $t \in [t_n, \bar{t}_n]$, $E(\psi) < E(u(t))$. Then from (5.6) with t_0 being replaced by t_n, and t being replaced by $\bar{t}_n$ we deduce that

$$\int_{t_n}^{\bar{t}_n} \| u_t \|_{\mathrm{X}}\, d\tau \leq C_\theta(E(u(t_n)) - E(\psi))^\theta < \frac{\varepsilon}{2}. \tag{5.12}$$

Thus we have

$$\| u(\bar{t}_n) - \psi \|_{\mathrm{X}} \leq \| u(t_n) - \psi \|_{\mathrm{X}} + \int_{t_n}^{\bar{t}_n} \| u_t \|_{\mathrm{X}}\, d\tau < \varepsilon, \tag{5.13}$$

which implies that when $n \to +\infty$,

$$u(\bar{t}_n) \to \psi \qquad \text{in X}.$$

Since $\bigcup_{t \geq \delta} u(t)$ is relatively compact in $H^2(\Omega)$, there exists a subsequence of $\{u(\bar{t}_n)\}$, still denoted by $\{u(\bar{t}_n)\}$ converging to ψ in $H^2(\Omega)$. Namely, when n is sufficiently large, we have

$$\| u(\bar{t}_n) - \psi \|_{H^2(\Omega)} < \beta_0,$$

which contradicts the definition of $\bar{t}_n$ that $\|u(\bar{t}_n, \cdot) - \psi\|_{H^2(\Omega)} = \beta_0$.

Part II. Convergence Rate

For $t \geq t_0$, it follows from Lemma 4.5 and (5.5) that

$$\frac{d}{dt}(E(u) - E(\psi)) + C\,(E(u) - E(\psi))^{2(1-\theta)} \leq 0. \tag{5.14}$$

As a result,

$$E(u(t)) - E(\psi) \le C(1+t)^{-1/(1-2\theta)}, \qquad \forall\, t \ge t_0. \tag{5.15}$$

Integrate (5.5) on (t,∞), where $t \ge t_0$, then we have

$$\int_t^\infty \|u_t\|_{\mathrm{X}} d\tau \le C(1+t)^{-\theta/(1-2\theta)}. \tag{5.16}$$

By adjusting the constant C properly, we obtain

$$\|u(t)-\psi\|_{\mathrm{X}} \le C(1+t)^{-\theta/(1-2\theta)}, \quad t \ge 0. \tag{5.17}$$

Based on this convergence rate we are able to get the same estimate for convergence rate in higher order norm by energy estimates and proper differential inequalities.
Next we proceed to estimate $\|u-\psi\|_V$.
It follows from (1.1)-(1.4) and the stationary problem (1.18) that

$$\begin{cases} \dfrac{d}{dt}(u-\psi) = \Delta\mu, \\ \mu = -\Delta(u-\psi) + f(u) - f(\psi), \end{cases} \tag{5.18}$$

with the boundary condition

$$\begin{cases} -\Delta_{\|}(u-\psi) + \partial_\nu(u-\psi) + (u-\psi) = \mu \\ (u-\psi)_t + \partial_\nu\mu + \mu = 0. \end{cases} \tag{5.19}$$

Using (5.18)(5.19), we take the inner product in $\mathcal{H}$ of $A^{-1}(u-\psi)_t$ with $(u-\psi)$ to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u-\psi\|_{\mathrm{X}}^2 + \|\nabla u - \nabla\psi\|^2 + \int_\Omega (f(u)-f(\psi))(u-\psi)dx \\ &\quad + \left\|\nabla_{\|}(u-\psi)\right\|_{L^2(\Gamma)}^2 + \|u-\psi\|_{L^2(\Gamma)}^2 \\ = &\ 0. \end{aligned} \tag{5.20}$$

On the other hand, by (5.18)(5.19) and taking the inner product in $\mathcal{H}$ of $(u-\psi)_t$ with μ, we have

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|\nabla u - \nabla\psi\|^2 + \int_\Omega F(u)dx - \int_\Omega f(\psi)udx + \frac{1}{2}\left\|\nabla_{\|}(u-\psi)\right\|_{L^2(\Gamma)}^2 \right. \\ &\left. +\frac{1}{2}\|u-\psi\|_{L^2(\Gamma)}^2\right) + \|\nabla\mu\|^2 + \|\mu\|_{L^2(\Gamma)}^2 \\ = &\ 0. \end{aligned} \tag{5.21}$$

Adding (5.20)(5.21) together, we have

$$\frac{d}{dt}\left(\frac{1}{2}\|u-\psi\|_{\mathrm{X}}^2 + \frac{1}{2}\|u-\psi\|_{L^2(\Gamma)}^2 + \frac{1}{2}\left\|\nabla_{\|}(u-\psi)\right\|_{L^2(\Gamma)}^2 + \frac{1}{2}\|\nabla u - \nabla\psi\|^2\right.$$

$$
\begin{aligned}
&+\int_\Omega F(u)dx-\int_\Omega F(\psi)dx+\int_\Omega f(\psi)\psi dx-\int_\Omega f(\psi)udx\Big)\\
&+\|\nabla u-\nabla\psi\|^2+\left\|\nabla_\|(u-\psi)\right\|^2_{L^2(\Gamma)}+\|u-\psi\|^2_{L^2(\Gamma)}+\|\nabla\mu\|^2+\|\mu\|^2_{L^2(\Gamma)}\\
= & -\int_\Omega(f(u)-f(\psi))(u-\psi)dx. \qquad (5.22)
\end{aligned}
$$

In what follows, we shall use the uniform bounds obtained in Lemma 2.1. Without loss of generality, we set $\delta=1$ in Lemma 2.1.

The Newton-Leibniz formula

$$
F(u)=F(\psi)+f(\psi)(u-\psi)+\int_0^1\int_0^1 f'(szu+(1-sz)\psi)(u-\psi)^2dsdz, \qquad (5.23)
$$

yields that

$$
\begin{aligned}
&\left|\int_\Omega F(u)dx-\int_\Omega F(\psi)dx+\int_\Omega f(\psi)\psi dx-\int_\Omega f(\psi)udx\right|\\
= & \left|\int_\Omega\int_0^1\int_0^1 f'(szu+(1-sz)\psi)(u-\psi)^2dsdzdx\right|\\
\le & \max_{s,z\in[0,1]}\|f'(szu+(1-sz)\psi)\|_{L^3}\|u-\psi\|^2_{L^3}\\
\le & C(\|\nabla u-\nabla\psi\|\|u-\psi\|+\|u-\psi\|^2)\\
\le & \frac{1}{4}\|\nabla u-\nabla\psi\|^2+C\|u-\psi\|^2, \quad t\ge 1. \qquad (5.24)
\end{aligned}
$$

and in a similar way, we have

$$
\begin{aligned}
&\left|\int_\Omega(f(u)-f(\psi))(u-\psi)dx\right|\\
= & \left|\int_\Omega\int_0^1 f'(su+(1-s)\psi)(u-\psi)^2dsdx\right|\\
\le & \frac{1}{4}\|\nabla u-\nabla\psi\|^2+C\|u-\psi\|^2, \quad t\ge 1. \qquad (5.25)
\end{aligned}
$$

Let

$$
\begin{aligned}
y_1(t) = & \frac{1}{2}\|u-\psi\|^2_{\mathrm{X}}+\frac{1}{2}\|u-\psi\|^2_{L^2(\Gamma)}+\frac{1}{2}\left\|\nabla_\|(u-\psi)\right\|^2_{L^2(\Gamma)}+\frac{1}{2}\|\nabla u-\nabla\psi\|^2\\
&+\int_\Omega F(u)dx-\int_\Omega F(\psi)dx+\int_\Omega f(\psi)\psi dx-\int_\Omega f(\psi)udx \qquad (5.26)
\end{aligned}
$$

(5.24) indicates that there exist constants $C_1, C_2>0$ such that

$$
y_1(t)\ge C_1\|u-\psi\|^2_V-C_2\|u-\psi\|^2, \quad t\ge 1. \qquad (5.27)
$$

On the other hand,

$$
\|u-\psi\|^2 \le C\|u-\psi\|_V\|u-\psi\|_{\mathrm{X}}
$$

$$\leq \varepsilon\|u-\psi\|_V^2 + C(\varepsilon)\|u-\psi\|_{\mathrm{X}}^2. \tag{5.28}$$

From (5.17)(5.22)(5.24)(5.27)(5.28), after taking $\varepsilon > 0$ sufficiently small, we can see that there exists a constant $\gamma > 0$ such that

$$\frac{d}{dt}y_1(t) + \gamma y_1(t) \leq C\|u-\psi\|_{\mathrm{X}}^2 \leq C(1+t)^{-2\theta/(1-2\theta)}, \qquad t \geq 1. \tag{5.29}$$

As a result,

$$\begin{aligned}
y_1(t) &\leq y_1(1)e^{\gamma(1-t)} + Ce^{-\gamma t}\int_1^t (1+\tau)^{-2\theta/(1-2\theta)}d\tau \\
&\leq Ce^{-\gamma t} + Ce^{-\gamma t}\int_0^t (1+\tau)^{-2\theta/(1-2\theta)}d\tau \\
&\leq Ce^{-\gamma t} + Ce^{-\gamma t}\left(\int_0^{\frac{t}{2}} e^{\gamma\tau}(1+\tau)^{-2\theta/(1-2\theta)}d\tau + \int_{\frac{t}{2}}^t e^{\gamma\tau}(1+\tau)^{-2\theta/(1-2\theta)}d\tau\right) \\
&\leq Ce^{-\gamma t} + Ce^{-\gamma t}\left(e^{\frac{\gamma}{2}t}\int_0^{\frac{t}{2}} (1+\tau)^{-2\theta/(1-2\theta)}d\tau + C(1+t)^{-2\theta/(1-2\theta)}e^{\gamma t}\right) \\
&\leq C(1+t)^{-2\theta/(1-2\theta)}, \quad t \geq 1.
\end{aligned} \tag{5.30}$$

(5.27)(5.28)(5.30) imply that

$$\begin{aligned}
C_1\|u-\psi\|_V^2 &\leq y_1(t) + C_2\|u-\psi\|^2 \\
&\leq y_1(t) + C_2\varepsilon\|u-\psi\|_V^2 + C_2C(\varepsilon)\|u-\psi\|_{\mathrm{X}}^2.
\end{aligned} \tag{5.31}$$

Taking $\varepsilon > 0$ sufficiently small, it follows from (5.17)(5.30) that

$$\|u-\psi\|_V \leq C(1+t)^{-\theta/(1-2\theta)}, \qquad t \geq 1. \tag{5.32}$$

By the C^∞ regularity of the solution, we are able to get the estimate for convergence rate in higher order norm.

Differentiating (1.1)–(1.4) respect to time t respectively, we have

$$u_{tt} = \Delta\mu_t, \quad x \in \Omega, \tag{5.33}$$

$$\mu_t = -\Delta u_t + f'(u)u_t, \quad x \in \Omega, \tag{5.34}$$

$$u_{tt} + \partial_\nu\mu_t + \mu_t = 0, \quad x \in \Gamma, \tag{5.35}$$

$$\mu_t = -\Delta_\| u_t + \partial_\nu u_t + u_t, \quad x \in \Gamma. \tag{5.36}$$

Multiplying (5.34) by u_t and integrating by parts on Ω, using (1.3)(5.36), we get

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\mu\|^2 + \|\mu\|_{L^2(\Gamma)}^2\right) + \|\nabla u_t\|^2 + \int_\Gamma \left(\left|\nabla_\| u_t\right|^2 + u_t^2\right) dS$$

$$= \quad -\int_\Omega f'(u)u_t^2 dx. \tag{5.37}$$

Assumption (**F3**) yields that there is a certain positive constant $M_f \geq 1$ such that

$$f'(s) \geq -M_f, \qquad s \in \mathbb{R}. \tag{5.38}$$

Thus,

$$-\int_\Omega f'(u)u_t^2 dx \leq M_f \|u_t\|^2. \tag{5.39}$$

It follows from (1.1) that

$$\begin{aligned}
\|u_t\|^2 &= -\int_\Omega \nabla u_t \cdot \nabla \mu dx - \int_\Gamma \mu u_t dS - \|u_t\|^2_{L^2(\Gamma)} \\
&\leq \|\nabla u_t\|\|\nabla \mu\| + \|\mu\|_{L^2(\Gamma)}\|u_t\|_{L^2(\Gamma)} \\
&\leq \varepsilon\|\nabla u_t\|^2 + \varepsilon\|u_t\|^2_{L^2(\Gamma)} + \frac{1}{4\varepsilon}\|\nabla \mu\|^2 + \frac{1}{4\varepsilon}\|\mu\|^2_{L^2(\Gamma)},
\end{aligned} \tag{5.40}$$

In (5.40), taking

$$\varepsilon = \frac{1}{2M_f}, \tag{5.41}$$

it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla \mu\|^2 + \|\mu\|^2_{L^2(\Gamma)}\right) + \frac{1}{2}\|\nabla u_t\|^2 + \|\nabla_\| u_t\|^2_{L^2(\Gamma)} + \frac{1}{2}\|u_t\|^2_{L^2(\Gamma)} \\
\leq\ & C\left(\|\nabla \mu\|^2 + \|\mu\|^2_{L^2(\Gamma)}\right).
\end{aligned} \tag{5.42}$$

Multiplying (5.34) by u_{tt} and integrating by parts on Ω, using (5.33)(5.35)(5.36), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u_t\|^2 + \int_\Omega f'(u)u_t^2 dx + \|\nabla_\| u_t\|^2_{L^2(\Gamma)} + \|u_t\|^2_{L^2(\Gamma)}\right) \\
&\quad + \|\mu_t\|^2_{L^2(\Gamma)} + \|\nabla \mu_t\|^2 \\
=\ & \frac{1}{2}\int_\Omega f''(u)u_t^3 dx.
\end{aligned} \tag{5.43}$$

By (5.40) and Lemma 2.1, the righthand side of (5.43) can be estimated as follows

$$\begin{aligned}
&\left|\int_\Omega f''(u)u_t^3 dx\right| \\
\leq\ & C(|u|_{L^\infty})\|u_t\|^3_{L^3} \leq C\|u\|_{H^2}\left(\|\nabla u_t\|^{\frac{3}{2}}\|u_t\|^{\frac{3}{2}} + \|u_t\|^3\right) \\
\leq\ & \frac{1}{8}\|\nabla u_t\|^2 + C\|u_t\|^6 + C\|u_t\|^3 \\
\leq\ & \frac{1}{8}\|\nabla u_t\|^2 + C\|u_t\|^2 \\
\leq\ & \frac{1}{4}\left(\|\nabla u_t\|^2 + \|u_t\|^2_{L^2(\Gamma)}\right) + C\|\nabla \mu\|^2 + C\|\mu\|^2_{L^2(\Gamma)}, \qquad t \geq 1.
\end{aligned} \tag{5.44}$$

Then (5.43) becomes

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u_t\|^2+\int_\Omega f'(u)u_t^2dx+\|\nabla_\parallel u_t\|^2_{L^2(\Gamma)}+\|u_t\|^2_{L^2(\Gamma)}\right)\\
&\quad+\|\mu_t\|^2_{L^2(\Gamma)}+\|\nabla\mu_t\|^2\\
\le\ &\frac{1}{8}\left(\|\nabla u_t\|^2+\|u_t\|^2_{L^2(\Gamma)}\right)+C\|\nabla\mu\|^2+C\|\mu\|^2_{L^2(\Gamma)},\qquad t\ge 1.
\end{aligned}
\tag{5.45}
$$

Multiplying (5.45) by $\varepsilon_1\in(0,1]$ and adding the resultant to (5.42), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\mu\|^2+\|\mu\|^2_{L^2(\Gamma)}+\varepsilon_1\|\nabla u_t\|^2+\varepsilon_1\int_\Omega f'(u)u_t^2dx+\varepsilon_1\left\|\nabla_\parallel u_t\right\|^2_{L^2(\Gamma)}\\
&+\varepsilon_1\|u_t\|^2_{L^2(\Gamma)}\Big)+\frac{1}{4}\left(\|\nabla u_t\|^2+\|u_t\|^2_{L^2(\Gamma)}\right)+\left\|\nabla_\parallel u_t\right\|^2_{L^2(\Gamma)}\\
&+\varepsilon_1\|\mu_t\|^2_{L^2(\Gamma)}+\varepsilon_1\|\nabla\mu_t\|^2\\
\le\ &C^*\left(\|\nabla\mu\|^2+\|\mu\|^2_{L^2(\Gamma)}\right),\qquad t\ge 1.
\end{aligned}
\tag{5.46}
$$

Let

$$
y_2(t)=\|\nabla\mu\|^2+\|\mu\|^2_{L^2(\Gamma)}+\varepsilon_1\|\nabla u_t\|^2+\varepsilon_1\int_\Omega f'(u)u_t^2dx+\varepsilon_1\left\|\nabla_\parallel u_t\right\|^2_{L^2(\Gamma)}+\varepsilon_1\|u_t\|^2_{L^2(\Gamma)} \tag{5.47}
$$

It follows from Lemma 2.1 that

$$
y_2(t)\le C,\qquad t\ge 1. \tag{5.48}
$$

Taking

$$
\varepsilon_1=\frac{1}{M_f^2}, \tag{5.49}
$$

we can deduce from (5.39)–(5.41) that

$$
y_2(t)\ge\frac{1}{2}\left(\|\nabla\mu\|^2+\|\mu\|^2_{L^2(\Gamma)}\right)+\varepsilon_1\left(\frac{1}{2}\|\nabla u_t\|^2+\left\|\nabla_\parallel u_t\right\|^2_{L^2(\Gamma)}+\frac{1}{2}\|u_t\|^2_{L^2(\Gamma)}\right). \tag{5.50}
$$

Now we take $\kappa>0$ such that

$$
\kappa(1+C^*)\le\frac{1}{2}. \tag{5.51}
$$

Next, we multiply (5.46) by κ and add the resultant to (5.22), then (5.29)(5.25)(5.32) yield that there exists a constant $\widetilde{\gamma}>0$ such that

$$
\frac{d}{dt}[y_1(t)+\kappa y_2(t)]+\widetilde{\gamma}[y_1(t)+\kappa y_2(t)]\le C\|u-\psi\|^2\le C(1+t)^{-2\theta/(1-2\theta)},\quad t\ge 1. \tag{5.52}
$$

Similar to (5.30), we have

$$
y_1(t)+\kappa y_2(t)\le C(1+t)^{-2\theta/(1-2\theta)},\quad t\ge 1. \tag{5.53}
$$

Hence, from (5.27)(5.30)(5.32)(5.53) we know

$$y_2(t) \le C(1+t)^{-2\theta/(1-2\theta)}, \quad t \ge 1, \tag{5.54}$$

which together with (5.50) gives the following

$$\|\mu\|_{H^1(\Omega)} + \|u_t\|_{H^1(\Omega)} + \|u_t\|_{H^1(\Gamma)} \le C(1+t)^{-\theta/(1-2\theta)}, \quad t \ge 1. \tag{5.55}$$

By the elliptic estimate (see [17, Corollary A.1]),

$$\|u-\psi\|_{H^3(\Omega)} + \|u-\psi\|_{H^3(\Gamma)} \le C\left(\|\mu\|_{H^1(\Omega)} + \|f(u)-f(\psi)\|_{H^1(\Omega)} + \|\mu\|_{H^1(\Gamma)}\right). \tag{5.56}$$

Lemma 2.1 and Sobolev imbedding theorem imply that

$$\|f(u)-f(\psi)\|_{H^1(\Omega)} \le C\|u-\psi\|_V, \qquad t \ge 1. \tag{5.57}$$

On the other hand, from (5.2), the elliptic regularity theory and Sobolev imbedding theorem, we have

$$\|\mu\|_{H^1(\Gamma)} \le C\|\mu\|_{H^2(\Omega)} \le C\left(\|u_t\| + \|u_t\|_{H^{\frac{1}{2}}(\Gamma)}\right) \le C\|u_t\|_{H^1(\Omega)}. \tag{5.58}$$

As a result, we can conclude from (5.32), (5.55)–(5.58) that

$$\|u-\psi\|_{H^3(\Omega)} + \|u-\psi\|_{H^3(\Gamma)} \le C(1+t)^{-\theta/(1-2\theta)}, \quad t \ge 1. \tag{5.59}$$

Summing up, the proof of theorem 1.1 is completed.

Remark 5.1. *Following the same method, we can continue to get estimates of convergence rate in higher order norm.*

Remark 5.2. *We notice that, in order to get the convergence rate estimates* (5.32) (5.55) (5.59)*, we have to use the uniform bound for the solution in higher order norm, e.g. Lemma 2.1, which is not valid for $t = 0$. Thus, the constant C in* (5.32)(5.55)(5.59) *depends on δ in Lemma 2.1. More precisely, for any $\delta > 0$ we have*

$$\|u-\psi\|_{H^3(\Omega)} + \|u-\psi\|_{H^3(\Gamma)} + \|u_t\|_V \le C_\delta(1+t)^{-\theta/(1-2\theta)}, \quad \forall\ t \ge \delta. \tag{5.60}$$

the constant C_δ depends on $\|u_0\|_V$ and δ. Moreover,

$$\lim_{\delta\to 0^+} C_\delta = +\infty. \tag{5.61}$$

Acknowledgements: The author is indebted to Prof. Songmu Zheng for his enthusiastic help and encouragement. This work is partially supported by the NSF of China under grant No. 10631020 and Fudan Postgraduate Innovation Project under grant No. CQH 5928003.

References

[1] J.W. Cahn and E. Hilliard, Free energy of a nonuniform system, I, Interfacial free energy, J. Chem. Phys., **28** (1958), 258–367.

[2] R. Chill, E. Fašangová and J. Prüss, Convergence to steady states of solutions of the Cahn-Hilliard and Caginalp equations with dynamic boundary conditions, Math. Nachr. **279** (13&14) (2006), 1448–1462.

[3] C.M. Elliott and S. Zheng, On the Cahn-Hilliard equation, Arch. Rational Mech. Anal., **96** (1986), 339–357.

[4] C.G. Gal, A Cahn-Hilliard model in bounded domains with permeable walls, Math. Meth. Appl. Sci., **29** (2006), 2009–2036.

[5] C.G. Gal, Exponential attractors for a Cahn-Hilliard model in bounded domains with permeable walls, Electronic J. of Diff. Eqns., **2006**(143) (2006), 1–23.

[6] G.R. Goldstein, Derivation and physical interpretation of general boundary conditions, Adv. Differential Equations, **11**(4) (2006), 457–480.

[7] M. Grinfeld and A. Novick-Cohen, Counting stationary solutions of the Cahn-Hilliard equation by transversality arguments, Proc. Roy. Soc. Edinburgh, Sect. A, **125**(2) (1995), 351–370.

[8] A. Haraux, M. A. Jendoubi, Decay estimates to equilibrium for some evolution equations with an analytic nonlinearity, Asymptotic Analysis, **26**(1) (2001), 21–36.

[9] S.-Z. Huang and P. Takáč, Convergence in gradient–like systems which are asymptotically autonomous and analytic, Nonlinear Analysis, **46** (2001), 675–698.

[10] M. A. Jendoubi, A simple unified approach to some convergence theorem of L. Simon, J. Func. Anal., **153** (1998), 187–202.

[11] M.A. Jendoubi, Convergence of global and bounded solutions of the wave equation with linear dissipation and analytic nonlinearity, Journal of Differential Equations, **144** (1998), 302–312.

[12] R. Kenzler, F. Eurich, P. Maass, B. Rinn, J. Schropp, E. Bohl, and W. Dieterich, Phase separation in confined geometries: Solving the Cahn-Hilliard equation with generic boundary conditions, Computer Phys. Comm., **133** (2001), 139–157.

[13] S. Lojasiewicz, Une propri*été* topologique des sous-ensembles analytiques re*és*. Colloque Internationaux du C.N.R.S. **117**, Les equations aux deriv*ées* parielles (1963), 87–89.

[14] S. Lojasiewicz, Sur la geometrie semi- et sous-analytique . Ann. Inst. Fourier (Grenoble) **43** (1963), 1575–1595.

[15] S. Lojasiewicz, Ensemble semi-analytic . Bures-sur-Yvette: IHES, 1965.

[16] H. Matano, Convergence of solutions of one-dimensional semilinear parabolic equations, J. Math. Kyoto Univ., **18**(2) (1978), 221–227.

[17] A. Miranville and S. Zelik, Exponential attractors for the Cahn-Hilliard equation with dynamical boundary conditions, Math. Meth. in Appl. Sci., **28** (2005), 709–735.

[18] P. Mironescu and V. Radulescu, Nonlinear Sturm–Liouville type problems with a finite number of solutions, Matarom, Laboratoire d'Analyse Numerique, Universite Pierre et Marie Curie (Paris VI), **3** (1993), 54–67.

[19] A. Novick-Cohen and A. Peletier, Steady states of the one-dimensional Cahn-Hilliard equation. Proc. Roy. Soc. Edinburgh, Sect. A, **123**(6) (1993), 1071–1098.

[20] A. Novick-Cohen and L.A. Segel , Nonlinear aspects of the Cahn-Hilliard equation, Physica D, **10** (1984), 277–298.

[21] P. Polačik and K.P. Pybakowski, Nonconvergent bounded trajectories in semilinear heat equations, J. Diff. Eqs., **124** (1996), 472–494.

[22] P. Polačik and F. Simondon, Nonconvergent bounded solutions of semilinear heat equations on arbitrary domains, J. Diff. Eqs., **186** (2002), 586–610.

[23] J. Prüss, R. Racke and S. Zheng, Maximal regularity and asymptotic behavior of solutions for the Cahn–Hilliard Equation with dynamic boundary condition, Ann. Mat. Pura Appl., **185**(4) (2006), 627–648.

[24] R. Racke and S. Zheng, The Cahn-Hilliard equation with dynamical boundary conditions, Adv. Diff. Eqs., **8**(1) (2003), 83–110.

[25] P. Rybka and K.H. Hoffmann, Convergence of solutions to Cahn-Hilliard equation, Comm. PDEs, **24** (5&6), (1999), 1055–1077.

[26] L. Simon, Asymptotics for a class of nonlinear evolution equation with applicationa to geometric problems, Ann. of Math., **118** (1983), 525–571.

[27] L. Simon, *Theorems on Regularity and Singularity of Energy Minimizing Maps*, Lectures in Math. ETH Zürich, Birkhäuser Verlag, 1996.

[28] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics,* Appl. Math. Sci., **68**, Springer-Verlag, New York, 1988.

[29] H. Wu and S. Zheng, Convergence to equilibrium for the Cahn–Hilliard equation with dynamic boundary conditions, J. Differential Equations, **204** (2004), 511–531.

[30] H. Wu and S. Zheng, Convergence to equilibrium for the damped semilinear wave equation with critical exponent and dissipative boundary condition, Quart. Appl. Math., **64**(1) (2006), 167–188.

[31] H. Wu, Convergence to equilibrium for the semilinear parabolic equation with dynamic boundary condition, accepted by Adv. Math. Sci. Appl., 2007.

[32] H. Wu, M. Grasselli and S. Zheng, Convergence to equilibrium for a parabolic–hyperbolic phase–field system with Neumann boundary conditions, Math. Models and Methods Appl. Sci., **17**(1) (2007), 1–29.

[33] H. Wu, M. Grasselli and S. Zheng, Convergence to equilibrium for a nonlinear parabolic–hyperbolic phase–field system with dynamic boundary condition, J. Math. Anal. Appl., **329** (2007), 948–976.

[34] E. Zeidler, *Nonlinear functional analysis and its applications, I.*, Springer, New York, Berlin, Heidelberg, 1986.

[35] T.I. Zelenyak, Stabilization of solutions of boundary value problems for a second-order parabolic equation with one space variable, Differentsial'nye Uravneniya, 1968, 17–22.

[36] S. Zheng, Asymptotic behavior of solutions to the Cahn-Hilliard equation, Appl. Anal., **23** (1986), 165–184.